COL100 - Lec 7

1. Fibonacci Example

Math :

$\forall n \geq 2,\ F_n = F_{n-1} + F_{n-2}$ $\quad F_0 = 1, \quad F_1 = 1$

Recursive Alg

```
fun fib (n) = if n=0 orelse n=1
              then n
              else fib (n-1) + fib (n-2)
```

Proof of correctness : Show $F_n = fib(n)$

Iterative (via tail recursion) Alg

Observation : Let

$$a = F_0 = fib(0) = 0$$

$$b = F_1 = fib(1) = 1$$

then

$$fib(2) = fib(1) + fib(0)$$

$$= b + a$$

$$fib(3) = fib(2) + fib(1)$$

$$= (a+b) + b$$

$$= a + 2b$$

Goal

$$\text{fib\_iter}(n, \underline{a}, \underline{b}) = \begin{cases} 0 \ (= a) & \text{if } n = 0 \\ 1 \ (= b) & \text{Else if } n = 1 \\ \text{fib\_iter}(n-1,\ b,\ a+b) & \text{Else} \end{cases}$$

$a \quad b \quad a+b \quad a+2b \quad 2a+3b$

1^{st} step	a	is	a
	b	is	b
2^{nd} step	a	is	b
	b	is	a+b

Correctness

Is $\text{fib\_iter}(n, 0, 1) = F_n \quad [\forall n > 1]$

Basis :

$$F_0 = 0 = \text{fib\_iter}(n, 0, 1)$$
$$F_1 = 1 = \text{fib\_iter}(n, 0, 1)$$

I.H :

$$k \geq 1$$
$$F_k = \boxed{\text{fib\_iter}(k, 0, 1)}$$

I.S. :

$$\text{fib\_iter}(k+1, 0, 1)$$
$$= \text{fib\_iter}(k, 1, 1)$$

I.H. ??

can't apply

Let us try to get more intuition and understanding between

$$\text{fib\_iter}(n, a, b) \text{ and } F_n$$

$$\text{fib\_iter}(0, a, b) = a$$

$$\text{fib\_iter}(1, a, b) = b$$

$$\text{fib\_iter}(2, a, b) = a + b = aF_1 + bF_2$$

$$\text{fib\_iter}(3, a, b) = a + 2b = aF_2 + bF_3$$

$$\text{fib\_iter}(4, a, b) = 2a + 3b = aF_3 + bF_4$$

The pattern suggests

$$\text{fib\_iter}(n, a, b) = a F_{n-1} + b F_n \quad , \forall\, n > 1$$

<u>Basis</u> : $n = 2$

$$\text{Fib\_iter}(2, a, b) = a + b = a F_1 + b F_2$$

<u>I.H.</u> : $\text{fib\_iter}(k, a, b) = a F_{k-1} + b F_k$

<u>I.S.</u> : $\text{fib\_iter}(k+1, a, b)$

$= \text{fib\_iter}(k, b, a+b)$ [Apply def^n of fib_iter]

$$= b F_{k-1} + (a+b) F_k$$

$$= a F_k + b \left[F_{k-1} + F_k \right]$$

$$= a F_k + b F_{k+1}$$ [Appl'y defn of F_n]

Alternatively,

alternate invariant

$\forall j \geq 1$ $a = F_{j-1}$

$b = F_j$ then

$$\text{fib\_iter}(1, a, b) = b$$

and $\text{fib\_iter}(2, a, b) = \text{fib\_iter}(1, F_j, F_j + F_{j-1})$

$$= F_{j+1}$$

$\forall\, n \geq 1$ and $j \geq 1$

$$\text{fib\_iter}(n, F_{j-1}, F_j) = F_{n+j-1}$$

Try proving it!

Q: Time Complexity
[bound on the number of recursive unfoldings]

$$T(n) = \begin{cases} 0 & F_0, F_1 \\ 2 + T(n-1) + T(n-2) & \text{otherwise} \end{cases} \leq 2^{n-1}, \forall n > 2$$

2. Pascal's Triangle

— Numbers on the edges are all 1

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

— Each number inside the $\triangle$ is the sum of the two numbers above it.

Q: Compute Elements of the Pascal $\triangle$ recursively

<u>Observation</u>

What is the recurrence relation

- $Pascal(n, 0) = Pascal(n, n) = 1$
- $Pascal(n, d) = Pascal(n-1, d-1) + Pascal(n-1, d)$

  $0 < d < n$

Challenge: Construct an iterative version of Pascal's triangle computation!

3. Counting Change

- Given Rs 1/- [in general Rs a]
- Given infinite supply of 1 p, 5 p, 10 p, 25 p, 50 p coins [in general n kinds of coins]
- Write a recursive procedure to compute the number of ways to change any given money.

Observation

Number of ways to change Rs a/- using n kinds of coins =

- Number of ways to change amount Rs1 using all but one kind of coins

+

- Number of way to change amount a-d using all n kinds of coins
[d is the denomination of the first kind of coins]

- if amount $a = 0$ then only 1 way
- if amount $a > 0$ but $n = 0$ then 0 way

Eg: $a = 4$ $N = \{1, 2, 3\}$

$\{1, 1, 1, 1\} + \{1, 1, 2\}, + \{2, 2\} + \{1, 3\}$

- for each coin with denomination d
  - we include that coin in the solution and recur with remaining change $(a-d)$ with same number of coins
  - we exclude the current coin and recur for remaining coins

Eg: 100 in terms of 50 and 25

$$cc(100, 2)$$

$$\boxed{2}\quad cc(50, 2) \;+\; cc(100, 1)\quad \boxed{1}$$

$\{25, 25, 25, 25\}$

$\{50, 50\}$ $\{25, 25, 50\}$

$$cc(0, 2)_{\boxed{1}} + \overset{\boxed{1}}{cc(50, 1)}$$

$$cc(75, 1) + cc(100, 0)\quad \boxed{1}$$

$$\overset{\boxed{1}}{cc(25, 1)} + \overset{\boxed{0}}{cc(50, 0)}$$

$$\overset{\boxed{1}}{cc(50, 1)} + cc(75, 0)$$

$$\overset{\boxed{1}}{cc(25, 1)} + cc(50, 0)$$

$$\overset{\boxed{1}}{cc(0, 1)} + \overset{\boxed{0}}{cc(25, 0)}$$

$$\overset{\boxed{1}}{cc(0, 1)} + \overset{\boxed{0}}{cc(25, 0)}$$

Thus,

$$cc(a, n) = \begin{cases} 0 & \text{if } n = 0 \wedge a > 0 \\ 1 & \text{Else if } a = 0 \wedge n > 0 \\ cc(a-d, n) + cc(a, n-1) & \text{Else} \end{cases}$$

Time complexity?